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Inequalities and variational principles for turbulent transport

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Two inequalities are proposed for the purpose of bounding from below the mean shear $U_0'(z)$ in turbulent channel flow. The first of these inequalities pertains to the energetics of the boundary layer, and the second pertains to the logarithmic form of the asymptote to U_0. These inequalities imply a maximum value of von Kármán's constant, the numerical value of which lies between the measurements of Laufer (1951) and the value obtained from bulk discharge measurements in a pipe. The formalism, which contains no adjustable parameters, is then applied to the turbulent thermal convection problem, and a lower bound for the mean temperature $\overline{T}_0(z)$ is obtained. The minimum value of the latter at large distances from the boundary is in fair agreement with Townsend's (1959) measurements. Although the proposed inequalities have not been deduced from the equations of motion, they provide facts which may be useful in the search for new variational formulations of the turbulent transport problem.

1. Introduction

Figure 1 shows a wide rectangular channel of height D and length L which is connected to an infinite reservoir which is filled to a given level (head) with a liquid of density ρ and viscosity ν (cm^2/s). When $L \to \infty$ a laminar Poiseulle flow is realized, when $L \to 0$ a free Bernoulli discharge is realized, and the turbulent regime of present interest occurs when L is large (but finite) and when the overall Reynolds number $\tau^{\frac{1}{2}}D/\nu$ is very large. In the statistically steady state the mean wall stress τ (cm^2/s^2) is obtained, according to the momentum principle, by multiplying the pressure head in the reservoir and $D/2\rho L$. The horizontally averaged $U(z)$ in the centre of the channel is symmetric and satisfies the boundary conditions

$$U(0) = 0, \tag{1.1}$$

$$U'(0) = \tau/\nu \tag{1.2}$$

at the bottom $z = 0$. A fundamental problem is to determine the discharge

$$Q = 2\int_0^{\frac{1}{2}D} U\,dz = 2\int_0^{\frac{1}{2}D} dz U'(z)\,[\tfrac{1}{2}D - z] \tag{1.3}$$

and the velocity profile.

Malkus (1956) suggested that for a given τ the observed mean flow $U_0(z)$ has a smaller value of Q than any other statistically steady solution of the Navier–Stokes equations, the idea being that these solutions are non-unique and densely degenerate.

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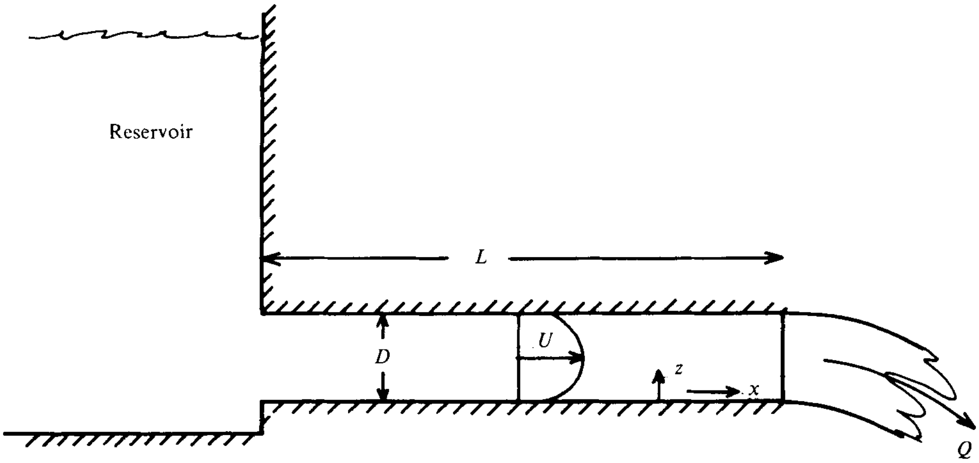

FIGURE 1. Schematic diagram for turbulent channel flow.

Malkus also had to introduce other controversial ideas for the constraints, one of these being the assumption that the (formal) substitution of $U_0(z)$ into the Orr–Sommerfeld equation would yield positive growth rates for some of the normal modes. Subsequent calculations by Reynolds & Tiedermann (1967), Gol'dshtik (1969), and others indicate that this assumption is not correct. The purpose of this investigation is to elucidate a set of constraints which are consistent with observations, and which can be used to test such variational principles as suggested by Malkus.

The proposed constraints are inequalities pertaining to the mean field, a relatively simple example of which is the well known fact that the curvature is negative, or

$$U''(z) \leqslant 0. \tag{1.4}$$

[There is no satisfactory dynamical explanation for this, and other types of turbulent flow (e.g. a free jet) do exhibit inflexion points.] The question arises as to whether other heuristic inequalities having more dynamical content can be elucidated, and for this purpose the following guidelines will be adopted. The rules for formulating the inequalities must be correct statements of fact which have some physical basis, so that it may be possible to generalize them to other turbulence problems (e.g. §5) without recourse to an experimental adjustment of parameters. The formalism will be inferred from plausible dynamical ideas, and the implications of the inequalities will then be compared with observations. No attempt is made in this paper to deduce the inequalities from the equations of motion.

The first inequality (§2) is a quantitative statement of the idea that the region of large shear near the wall is such that some two dimensional 'test perturbation' in the boundary layer is capable of releasing more energy than it dissipates. The plausible functional inequality (2.3) used to express this idea is phrased in the language of Orr (1907), and is verified by comparing its consequences with Laufer's (1954) measurements of U_0 near the wall. The new functional inequality, together with (1.1)–(1.4), describes a large class of shear profiles $U'(z)$, one of which is the observed U_0', and therefore the minimum of (1.3) gives a lower bound on the realized discharge.

The 'optimal' U for this first variational problem has no shear in the region beyond the wall boundary layer, whereas the observed U_0 has the well-known logarithmic boundary layer as described by von Kármán's constant K. Thus we proceed to a second inequality which will further restrict our class of U to those having a logarithmic asymptote with a *bounded* value of K. The inequalities (3.2) used for this purpose lead to a manifold of U', the envelope of which is assumed to bound the observed shear. The validity of this second assumption depends on the maximum value of K in the manifold, and comparison of theory and experiment is given in §4.

In order to demonstrate that the foregoing inequalities represent more than an exotic form of curve fitting, we have applied them without 'adjustment' (§5) to the problem of thermal turbulence (Bénard convection) at very high Rayleigh number. Here we have a thermal boundary layer and the straightforward generalization of our inequalities is shown to be consistent with the measurements by Townsend (1959) and Deardorff & Willis (1967). This first inequality provides a lower bound for the mean temperature gradient $\overline{T}_0'(z)$ for a given heat flux. The second inequality is related to the similarity law (Malkus 1963; Townsend 1959) for $\overline{T}_0 \propto z^{-1}$ at large distances from the lower boundary. The present theory gives a bound for the coefficient in this z^{-1} law which is in agreement with Townsend's measurement.

There are other approaches to the bounding problem, in which one uses only such constraints as are deducible from the equations of motion (Howard 1972). In the shear flow problem, for example, the main constraint is the mechanical energy integral

$$\nu \int_0^{\frac{1}{2}D} \overline{(\nabla \times \mathbf{V}_0)^2}\, dz = -\int_0^{\frac{1}{2}D} \overline{u_0 w_0}\; U_0'(z)\, dz, \tag{1.5}$$

where $\mathbf{V}_0 \equiv [u_0, v_0, w_0(x, y, z, t)]$ are the (x, y, z) components of fluctuating velocity, and a bar indicates a horizontal average. For the thermal convection problem there are two relevant energy integrals,

$$\left.\begin{aligned} \nu \int_0^{\frac{1}{2}D} \overline{(\nabla \times \mathbf{V}_0)^2}\, dz &= \int_0^{\frac{1}{2}D} g\alpha \overline{w_0 \mathscr{T}_0}\, dz, \\ \text{and} \qquad k \int_0^{\frac{1}{2}D} \overline{(\nabla \mathscr{T}_0)^2}\, dz &= -\int_0^{\frac{1}{2}D} \overline{w_0 \mathscr{T}_0}\, \overline{T}_0'(z)\, dz, \end{aligned}\right\} \tag{1.6}$$

where $\overline{T}_0(z)$ is the horizontally averaged temperature, $\mathscr{T}_0(x, y, z, t)$ is the fluctuating temperature, α is the thermal expansion coefficient, k is the thermal diffusivity, and D is the vertical separation between the two perfectly conducting boundaries. The rigorous bounds for the heat flux $F = -k\overline{T}_0'(0)$, and for Q, obtained by this kind of an approach are interesting, but obviously limited by the amount of information introduced.

2. An inequality pertaining to the energetics of the turbulent boundary layer

Our object is to construct a class of shear profiles $U'(z)$ which bound the observed profile $U_0'(z)$ from below, and attention is naturally directed to the broken line which is drawn in figure 2(*a*). This is constructed by drawing the tangent to the shear profile from the point $(0, \tau/\nu)$. The abscissa of the tangent point, denoted by $z_1(U')$, is obviously a function of the profile U', and we let $z_{10}(U_0')$ denote the tangent point for the *observed* profile. The point of intersection of the tangent with the z axis is denoted

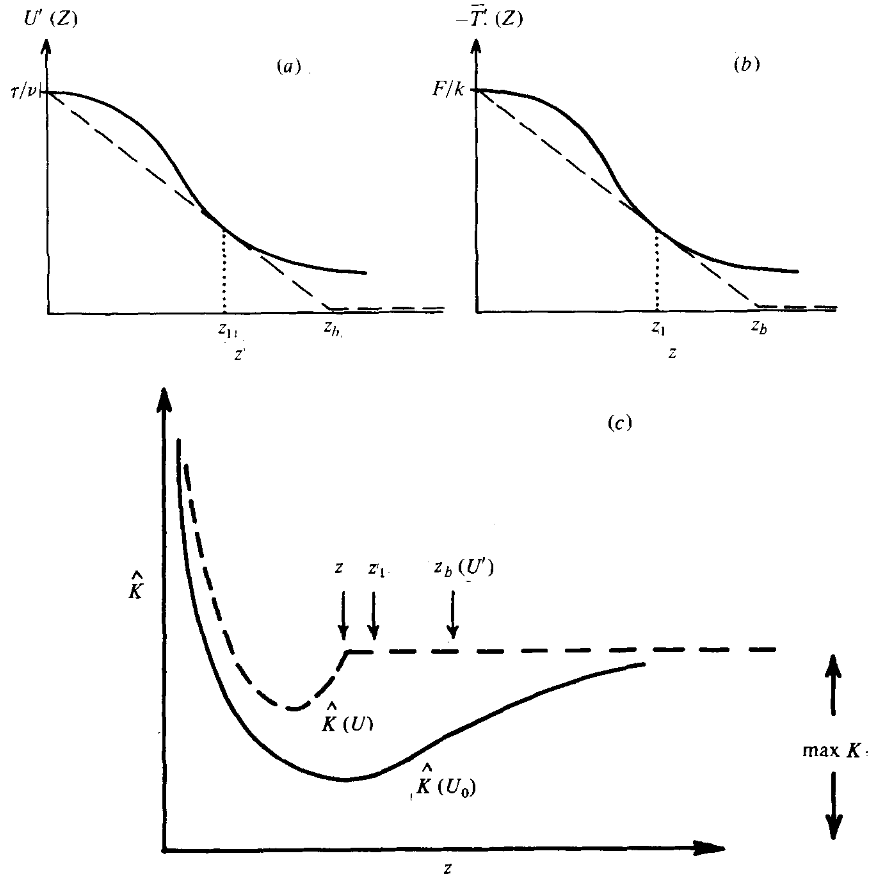

FIGURE 2. (a) Sketch of the mean shear $U'(z)$ as a function of distance from the boundary. The dashed line is tangential to the curve at $z = z_1$. (b) Sketch of the mean temperature gradient in turbulent thermal convection. (c) Geometrical meaning of the similarity inequality (3.9). The ordinate for these curves is the inverse shear function $\hat{K}(z) \equiv [dU(z)\,\tau^{-\frac{1}{2}}/d\ln z\tau^{\frac{1}{2}}\nu^{-1}]^{-1}$. The solid curve is a sketch of the observed $\hat{K}(U_0(z))$, the minimum point of which corresponds to the inflexion point in the plot of U_0 as a function of $\ln z$. The broken curve $\hat{K}(U(z))$, which has a constant value for z greater than $z_l < z_1(U')$, corresponds to a member U' of the M_2 manifold which bounds U_0' from below.

by $z_b(U')$, and we let $z_{b0} = z_b(U_0')$ denote the observed value of the boundary-layer width. The 'comparison profile' for $U_0'(z)$ is defined as that broken line curve (figure 2a) whose equation is:

$$U_{0c}'(z) = \begin{Bmatrix} (1-z/z_{b0})\,\tau/\nu, & z \leqslant z_{b0} \\ 0, & D-z_{b0} \geqslant z \geqslant z_{b0} \end{Bmatrix}. \tag{2.1}$$

The thickness z_{b0} of the boundary layer in (2.1) is obviously a well defined scale length of the observed profile U_0'. We also note that since $U_{0c}'(z) \leqslant U_0'(z)$, (1.1) implies $U_{0c}(z) \leqslant U_0(z)$.

The dynamical significance of the foregoing construction emerges from a consideration of (1.5), the right-hand side of which represents the generation of turbulent energy by the working of the Reynolds stress on the shear of the mean field. The left-hand side of (1.5) represents the equal amount of dissipation of the perturbation.

There are obviously many more perturbation fields besides the observed $\mathbf{V}_0$, such that $U_0'(z)$ will satisfy the same integral relation. If there exists any two dimensional velocity perturbations such that (1.5) can be satisfied, then we shall say that the given mean field U_0 is 'unstable in the sense of Orr (1907)'. This language, and some of the mathematical results of Orr will be found useful in formulating an inequality for the fully turbulent U_0. Otherwise there is no assumed connexion with Orr's problem, which is concerned with estimating when a *laminar* Poiseuille flow and a linear Couette flow becomes unstable. For each of these two laminar problems Orr computed the smallest possible value of the Reynolds number such that some two dimensional velocity perturbation would instantaneously generate more energy from the shear flow than that perturbation dissipates. In our turbulent proble.n reference is made to the observed mean flow, and (1.5) implies that there certainly are *three* dimensional test perturbations which are unstable in the sense of Orr. It is also a fact that there exist *two* dimensional 'test perturbations' which are unstable in the sense of Orr, because the overall Reynolds number is very large and because of the observed logarithmic shape of the boundary layer. Although this point is not necessary for the following development, a proof is given in appendix C.

Since the region of large U_0' near the wall is of crucial importance in the generation of turbulent kinetic energy, we now make the plausible assumption that the *comparison* profile (2.1) is also unstable in the sense of Orr. This means that there exists a two dimensional test perturbation

$$\mathbf{V}_c \equiv (u_c, 0, w_c) \equiv (-\partial\psi/\partial z, 0, \partial\psi/\partial x) \tag{2.2}$$

which generates as much energy from U_{0c}' as that perturbation dissipates, i.e.

$$1 = \frac{\nu \int_0^{\frac{1}{2}D} \overline{(\nabla \times \mathbf{V}_c)^2}\, dz}{-\int_0^{\frac{1}{2}D} \overline{u_c w_c}\, U_{0c}'\, dz} = \frac{\nu \int_0^{z_{b0}} \overline{(\nabla \times \mathbf{V}_c)^2}\, dz + \nu \int_{z_{b0}}^{\frac{1}{2}D} \overline{(\nabla \times \mathbf{V}_c)^2}\, dz}{\int_0^{z_{b0}} -\overline{u_c w_c}\, U_{0c}'\, dz}. \tag{2.3}$$

We shall require that the test perturbation (2.2) satisfies the no-slip boundary conditions at $z = 0$, and we stipulate that $\mathbf{V}_c$ vanishes at large distances $z \gg z_{b0}$ from the boundary layer. We will also require that the test perturbation have zero stress in the region where $U_{0c}'(z) = 0$, i.e. $\overline{u_c w_c} = 0$ for $z \geqslant z_{b0}$. Furthermore (u_c, w_c) will be continuous across z_{b0}, and will be negatively correlated where $U_{0c}' > 0$. No other conditions will be placed on $\mathbf{V}_c$, and it is emphasized that the introduction of this two dimensional 'test perturbation' does not mean that the turbulence is assumed to be two dimensional. Although U_{0c} and $\mathbf{V}_{0c}$ are formal *constructions*, the physical content of the assumption (2.3) will emerge from the fact that z_{b0} is a real and physically observable quantity. We shall see that (2.3) implies that the boundary layer Reynolds number cannot be 'too small'; for otherwise (2.3) could not be satisfied because the numerator on the right-hand side of (2.3) would exceed the denominator for all possible values of the 'dummy variable' $\mathbf{V}_c$.

It is obvious that for a given z_{b0} the right-hand side of (2.3) can be made as large as we please by merely increasing the horizontal wave number of $\mathbf{V}_c$. Therefore the truth of (2.3) can be established by merely exhibiting one function $\mathbf{V}_c(x, z)$ which makes the right-hand side of (2.3) *less* than unity, when the *observed* value z_{b0} is used.

The demonstration of this fact is deferred to appendix A, so as not to interfere with the development of the argument, and the consequences are as follows.

Since the last term in the numerator of (2.3) is positive, it follows that if (2.3) is correct then there must also exist a $\mathbf{V}_c$ (different from the previous one) such that

$$1 = \frac{\nu \int_0^{z_{b0}} \overline{(\nabla \times \mathbf{V}_c)^2}\, dz}{\int_0^{z_{b0}} (-\overline{u_c w_0})\, U'_{0c}\, dz}; \tag{2.4}$$

$$u_c(x,0) = w_c(x,0) = 0, \tag{2.4a}$$

$$\overline{u_c(x, z_{b0})\, w_c(x, z_{b0})} = 0, \tag{2.4b}$$

and

$$-\frac{\overline{u_c w_c}}{(\overline{u_c^2})^{\frac{1}{2}}\,(\overline{w_c^2})^{\frac{1}{2}}} > 0, \quad 0 \leqslant z < z_{b0}; \tag{2.4c}$$

where (2.4*a*–*c*) are the boundary conditions previously stated. Next we note that for any $\mathbf{V}_c$ the replacement of U'_{0c} by $U'_0 > U'_{0c}$ *lowers* the value right-hand side of (2.4), and thus it follows that there must exist some $\mathbf{V}_c$ (different from the previous ones) which will satisfy

$$1 = \frac{\nu \int_0^{z_{b0}} \overline{(\nabla \times \mathbf{V}_c)^2}\, dz}{\int_0^{z_{b0}} (-\overline{u_c w_c})\, U'_0\, dz}. \tag{2.5}$$

Next, we replace U'_0 by any function U' satisfying (1.1), (1.2) and (1.4); and we replace z_{b0} by the $z_b(U')$ obtained from this U' by means of the tangent construction (figure 2*a*). This yields a manifold $M_1(U')$ of profiles which satisfy (2.4*a*–*c*) and

$$1 = \frac{\nu \int_0^{z_b} \overline{(\nabla \times \mathbf{V}_c)^2}\, dz}{\int_0^{z_b} (-\overline{u_c w_c})\, U'(z)\, dz} \tag{2.6}$$

with one of the members of the manifold being U'_0. It then follows that the observed discharge Q_0 must be greater than that computed from the minimum value of (1.4), or

$$Q_0 > 2 \min_{M_1} \int_0^{z_b} U'(z)\, [\tfrac{1}{2}D - z]\, dz. \tag{2.7}$$

The implications of the foregoing become clearer if we introduce the streamfunction (2.2) for $\mathbf{V}_c$, and make the result non-dimensional by using z_b, as the unit of length and τ/ν as the unit of shear, i.e. let

$$s(\zeta) = (\tau/\nu)^{-1}\, U'(z), \quad \zeta = z/z_b, \quad \xi = x/z_b, \quad \zeta_1 = z_1/z_b; \tag{2.8}$$

where ζ_1 is the non-dimensional abscissa of the tangent point. Thus (2.6) becomes

$$\frac{\nu^2}{\tau z_b^2} \frac{\int_0^1 \overline{(\nabla^2 \psi)^2}\, d\zeta}{\int_0^1 \overline{\psi_\xi \psi_\zeta}\, s(\zeta)\, d\zeta} = 1. \tag{2.9}$$

The constraints on the non-dimensional shear profile are

$$s(0) = 1, \quad s(\zeta_1) = 1-\zeta_1, \quad s'(\zeta_1) = -1, \quad s'(\zeta) \leqslant 0, \tag{2.10}$$

$$1-\zeta \leqslant s(\zeta) \leqslant 1; \tag{2.11}$$

and the boundary conditions become:

$$\psi(\xi, 0) = 0 = \psi_\zeta(\xi, 0); \tag{2.12}$$

$$\overline{\psi_\xi(\xi, 1)\,\psi_\zeta(\xi, 1)} = 0; \tag{2.13}$$

$$\overline{\psi_\xi \psi_\zeta}/(\overline{\psi_\xi^2})^{\frac{1}{2}}(\overline{\psi_\zeta^2})^{\frac{1}{2}} > 0, \quad 0 \leqslant \zeta < 1. \tag{2.14}$$

When (2.7) is made non-dimensional, and when the small term of order $\nu/\tau^{\frac{1}{2}}D \ll 1$ is discarded we get

$$\frac{Q_0}{\tau^{\frac{1}{2}}D} > \min_{\psi, s} \int_0^1 d\zeta s(\zeta) \left[\frac{\int_0^1 \overline{(\nabla^2\psi)^2}\, d\zeta}{\int_0^1 \overline{\psi_\xi \psi_\zeta} s(\zeta)\, d\zeta}\right]^{\frac{1}{2}}, \tag{2.15}$$

where (2.9) has been used to eliminate z_b.

The mathematical results obtained by Orr (1907) imply that the right-hand side of (2.15) is a definite number, the calculation of which is relegated to appendix A, so as not to distract from the line of argumentation. The value of the right-hand side of (2.15) corresponds to a particular 'drag coefficient', and the optimizing U is found to give zero shear for $z \geqslant z_b$. This optimal U' having zero shear is one of the members of a manifold $M_1(U')$ of profiles, the lower envelope $U'_{e1}(z)$ of which is guaranteed to bound the observed U'_0 at all z.

We now recognize that this 'optimal' U is unrealistic and physically inconsistent (momentum-wise) with the downstream pressure gradient which exists at *all* z in the channel. Therefore another constraint will be added to remove those profiles having this unrealistic region of zero shear above the boundary layer.

3. Similarity inequality

The most general and best established similarity assumption for turbulent channel and pipe flow is the law of the wall, which states that if z, ν and the pressure gradient are held constant, then as $D \to \infty$, $U_0(z)$ approaches a finite limit profile $U_{0w}(z)$. The law of the wall does not apply to the 'defect region' $z/D = O(1)$, and as $D \to \infty$ the maximum velocity $U_0(\frac{1}{2}D)$ increases without bound. For reasons of continuity it then follows that

$$U_{0w}(z) = \int_0^z \frac{dU_{0w}}{dz}\, dz \tag{3.0}$$

must also increase without bound when we let $z \to \infty$ in the law of the wall. This means that U'_{0w} must be sufficiently large so that the integral in (3.0) diverges as $z \to \infty$. Thus the asymptotic U'_{0w} must exceed (for example) a constant multiple of the function $z^{-1}(\ln z\tau^{\frac{1}{2}}\nu^{-1})^{-2}$ since the integral of this function converges at $z \to \infty$. One could therefore impose this function as a bound on U'_{0w} at $z \to \infty$, but for obvious empirical reasons we want to introduce a slightly stronger form of constraint which will give us the well known logarithmic law. We therefore make the further similarity assumption that the ratio of the shear (or any higher derivative) at z to the shear at az depends only on a and not on z (provided $a = O(1)$ and $\tau^{\frac{1}{2}}z/\nu \gg 1$). This functional relation, when differentiated with respect to z, implies a power law form for $U_{0w}\alpha z^{-\gamma}$ with the exponent γ being independent of z. By substituting this power law into (3.0), and by requiring the latter integral to diverge we obtain the inequality $\gamma \leqslant 1$. Thus U'_{0w} must be bounded below at $z \to \infty$ by a curve which is proportional to z^{-1}, and $U_{0w}(z)$ must increase at least as rapidly as $\ln z$.

The logarithmic constraint on the new manifold will be introduced by requiring the class of velocity profiles in (2.6) to satisfy

$$\frac{zU''(z)}{U'(z)} = -1 \quad \text{for} \quad z \geqslant z_l \tag{3.1}$$

and the central problem is what to *do* about the parameter z_l. Without introducing any additional constraints or experimentally adjustable parameters, we will consider each of the three possibilities:

$$z_l \geqslant z_b(U') \quad \text{or} \quad z_b \geqslant z_l > z_1 \quad \text{or} \quad 0 < z_l \leqslant z_1(U'). \tag{3.2}$$

Each of these inequalities, together with (3.1) and the previous conditions on M_1, specifies a new submanifold of profiles U', and in the absence of sufficient reason to the contrary, we assume that the envelope of each bounds the observed U'_{0w} from below. In order to verify this major assumption we will compare the envelope for each of the three problems with the observed U'_{0w}.

If K denotes the constant of integration for (3.1), then

$$U'(z) = \frac{\tau^{\frac{1}{2}}}{Kz} \tag{3.3}$$

and by using (2.8) this can be written in the non-dimensional form

$$s(\zeta) = \frac{\nu}{\tau^{\frac{1}{2}} z_b}\left(\frac{1}{K\zeta}\right), \quad \zeta \geqslant \zeta_l \equiv z_l/z_b, \tag{3.4}$$

and

$$\frac{\zeta s'(\zeta)}{s(\zeta)} = -1. \tag{3.5}$$

When (3.4) is evaluated at ζ_l, and when (2.9) is used to eliminate z_b we get

$$K^2 = \frac{1}{\zeta_l^2 s^2(\zeta_l)} \frac{\int_0^1 \overline{\psi_\xi \psi_\zeta} s(\zeta)\, d\zeta}{\int_0^1 \overline{(\nabla^2\psi)^2}\, d\zeta}. \tag{3.6}$$

At this point we may note that the profiles in M_1 (§2) are really degenerate members of (3.3) which correspond to the values $K = \infty$ and $z_l = z_b$. Therefore attention is directed to the question as to whether (3.6) has a finite maximum.

Consider the first inequality in (3.2), i.e. $\zeta_l \geqslant 1$. The only constraints (2.10) on $s(\zeta)$ apply to the interval $\zeta \leqslant 1$, and thus it is clear that $s(\zeta_l)$ may be made as small as we please. Thus (3.6) has no upper bound for this submanifold, and the latter has an envelope which is identical to that which was found for M_1 in §2. Therefore the assumption made following (3.2) is trivially correct for the case $z_l \geqslant z_b$.

In the remaining two cases $\zeta_l \leqslant 1$ and (3.4) implies $\zeta_l s(\zeta_l) = s(1)$. Therefore (3.6) becomes

$$K^2 = \frac{1}{s^2(1)} \int_0^1 \overline{\psi_\xi \psi_\zeta} s(\zeta)\, d\zeta \Big/ \int_0^1 \overline{(\nabla^2 \psi)^2}\, d\zeta. \tag{3.7}$$

Now consider the second inequality in (3.2), i.e. $1 \geqslant \zeta_l > \zeta_1$. In this case we may still let $s(1) \to 0$ with $\zeta_l \to 1$ and $\zeta_1 \to 1$, and thus we conclude that K also has no finite limit for the second submanifold, the envelope of which is also U_{e1}. We see that these two submanifolds really introduce no new information, and the desired finite logarithmic profile is 'not enforceable' by such weak inequalities as we are using.

Now consider the last inequality in (3.2), or the manifold M_2 defined by

$$M_2: \left\{ \begin{array}{l} M_1 \\ \dfrac{zU''(z)}{U'(z)} = -1 \quad \text{for} \quad z \geqslant z_l, z_l \leqslant z_1(U') \end{array} \right\} \tag{3.8}$$

and for which we have assumed

$$U'_{0w}(z) \geqslant U'_{e2}(z), \tag{3.9}$$

where U'_{e2} is the lower envelope of (3.8). A geometrical interpretation of this inequality is given in figure 2(*c*).

For this case we may evaluate (3.5) at $\zeta = \zeta_1 \geqslant \zeta_l$ and thus we have

$$\zeta_1 s'(\zeta_1)/s(\zeta_1) = -1.$$

The substitution of the values of $s(\zeta_1), s'(\zeta_1)$ from (2.10) then gives $\zeta_1/1-\zeta_1 = 1$, and consequently

$$\zeta_1 = \tfrac{1}{2}, \quad s(\zeta_1) = \tfrac{1}{2}, \quad s(\zeta) = \frac{\zeta_1 s(\zeta_1)}{\zeta} = \frac{1}{4\zeta}, \quad \zeta > \tfrac{1}{2}. \tag{3.10}$$

When (3.4) is evaluated at ζ_1 the expression for the boundary layer Reynolds number becomes

$$\frac{\tau^{\frac{1}{2}} z_b}{\nu} = \frac{4}{K}. \tag{3.11}$$

From (3.10) we have $s(1) = \frac{1}{4}$, and (3.7) then becomes

$$\left(\frac{K}{4}\right)^2 = \frac{\displaystyle\int_0^1 \overline{\psi_\xi \psi_\zeta} s(\zeta)\, d\zeta}{\displaystyle\int_0^1 \overline{(\nabla^2 \psi)^2}\, d\zeta}. \tag{3.12}$$

The form of s is unconstrained for $\zeta < \frac{1}{2}$, except for

$$1 \geqslant s(\zeta) \geqslant 1 - \zeta, \quad \zeta \leqslant \tfrac{1}{2} \tag{3.13}$$

and the function ψ is unconstrained except for the previously-stated boundary conditions at $\zeta = 0$ and $\zeta = 1$. As we shall see, these conditions ensure that the maximum value of (3.12) exists. Thus the last inequality in (3.2) is the only one which eliminates those profiles having zero shear beyond z_b and which provides new information. It remains, however, to establish the truth of (3.9).

For $z \gg \nu/\tau^{\frac{1}{2}}$, $U'_{e2}(z)$ is given by the value of (3.3) for which K is a maximum, and therefore we proceed to maximize (3.12) with respect to (s, ψ). From (3.10) and (3.12) we see that the largest s is given by

$$s(\zeta) \leqslant s_d \equiv \begin{cases} 1 & \text{for} \quad 0 < \zeta < \frac{1}{2} \\ \dfrac{1}{4\zeta} & \text{for} \quad \frac{1}{2} < \zeta \end{cases} \tag{3.14}$$

and therefore the largest value of (3.12) is

$$\max_{\psi, s} \left(\frac{K}{4}\right)^2 = \max_{\psi} \frac{\int_0^1 \overline{\psi_\xi \psi_\zeta} \, s_d(\zeta)\, d\zeta}{\int_0^1 \overline{(\nabla^2\psi)^2}\, d\zeta}. \tag{3.15}$$

This reduces the joint (ψ, s) variational problem to an ordinary one (with respect to ψ). By utilizing the mathematical results of Orr (1907), it is shown in appendix B that the numerical value of (3.15) is slightly larger than

$$\max K \simeq 0{\cdot}362. \tag{3.16}$$

The 'optimal' shear profile (3.14) associated with $\max K$ has a discontinuity at $\zeta_1 = \frac{1}{2} = \zeta_l$, or at $z = 2\nu/K\tau^{\frac{1}{2}}$, and therefore the 'optimal' velocity profile is

$$U(z) = \begin{cases} \tau z/\nu & z \leqslant 2\nu/K\tau^{\frac{1}{2}} = 5{\cdot}5\nu/\tau^{\frac{1}{2}}, \\ 5{\cdot}5\tau^{\frac{1}{2}} + \dfrac{\tau^{\frac{1}{2}}}{K} \ln \dfrac{\tau^{\frac{1}{2}} z}{5{\cdot}5\nu}, & z \geqslant 5{\cdot}5\nu/\tau^{\frac{1}{2}}. \end{cases} \tag{3.17}$$

Although the envelope $U'_{e2}(z)$ is determined by (3.16) and (3.3) for large z, the envelope at small z is determined by the smallest value of

$$|U''(0)| = (\tau/\nu)\, z_b^{-1}\, |s'(0)| = \tau^{\frac{3}{2}}\nu^{-2}(K/4)\, |s'(0)|.$$

Since $|s'(0)| \leqslant 1$ and $K \leqslant 0{\cdot}37$ we then have $|U''_{e2}(0)| \leqslant 0{\cdot}09\tau^{\frac{3}{2}}\nu^{-2}$.

4. Comparison with experiment and the significance of max K

In appendix A we show that the first of our two inequalities (2.3) agrees with experiment, and we now consider the second inequality (3.9). Since the envelope U'_{e2} is less than the derivative of (3.17), it is sufficient to show that the latter is less than or equal to the observed $U'_0(z)$.

For small values of z we refer to Laufer's (1954) pipe flow measurements which start at $\tau^{\frac{1}{2}}z/\nu = 3$ and which indicate a layer of constant shear extending up to $\tau^{\frac{1}{2}}z/\nu \simeq 5{\cdot}5$. When this is compared with (3.17), and when the last paragraph of §3 is taken into account, we conclude that our second inequality is correct for $\tau^{\frac{1}{2}}z/\nu \leqslant 5{\cdot}5$. For larger values of z the observed U'_0 decreases slowly, whereas (3.17) has a discontinuous decrease in slope, and therefore our inequality is also correct for $\tau^{\frac{1}{2}}z/\nu$ somewhat greater than 5·5. For still larger values of z we refer to the plot of $U_0(z)\tau^{-\frac{1}{2}}$ as a function of $\log z\tau^{\frac{1}{2}}/\nu$ in figure 7 of the channel flow experiment of Laufer (1951). From the asymptotic slope of the log region of this data we obtain $K = 0{\cdot}35$ as the approximate value of the von Kármán constant. Since this number is less than (3.16) (which in turn is less than $\max K$) we conclude that our second inequality agrees with Laufer's data over the entire range of the law of the wall which occurs in that experiment. Although that range may not be sufficient to give the precise value of the asymptotic slope (i.e. von Kármán's constant) it is most relevant for assessing the validity of our second inequality since (3.6) is evaluated at z_l, whereas the behaviour of the shear at very large $\tau^{\frac{1}{2}}z/\nu$ does not *explicitly* enter into our calculation. The significance of this remark appears below, where we consider other experimental determinations of von Kármán's constant.

The law of the wall for flow over a flat plate (Kline *et al.* 1967) overlaps Laufer's data provided the *measured* local value of τ is used, whereas a slightly larger value ($K = 0{\cdot}4$) of von Kármán's constant is obtained when Clauser's indirect method of reducing the data is used. This has been criticized by Kline *et al.* (1967) and thus it appears that the best laboratory evidence for the higher value of von Kármán's constant comes from correlations of the *bulk* discharge in a pipe with the measured pressure gradient. These measurements give $K = 0{\cdot}407$ (Townsend 1976, p. 149), and since this exceeds (3.16) the question arises as to whether or not our *second* inequality is correct.

Although the author is not in a position to fully evaluate the reason for the small difference in the experimental values of K cited above, it seems that the Laufer and Kline *et al.* data introduce a bias in favour of the lower portion of the logarithmic region, whereas the bulk discharge value of K gives greater weight to the upper portion of the log and defect layers which contribute most to the measured discharge. On the theoretical side a source of small error is due to the fact that (3.16) is only a lower bound [appendix, equation (B 4)] and a sharper determination is in order. My estimate (not included herein) is that $\max K = 0{\cdot}37 \pm 0{\cdot}01$, and it would be surprising if the precise calculation exceeded 0·4. A more satisfactory resolution of the small discrepancy in question is given by the following considerations, which also show how the present theory can be extended in order to incorporate more of the physics and to obtain a better bound for the entire profile.

The problem of bounding $U_0'(z)$ from below is obviously equivalent to bounding from above the 'von Kármán function' (see figure 2*c*):

$$\hat{K}(z) \equiv \left(\frac{dU_0(z)\,\tau^{-\frac{1}{2}}}{d\ln z\tau^{\frac{1}{2}}\nu^{-1}}\right)^{-1}.$$

From the previously cited measurements we know that this function decreases like z^{-1} for $z = 0$, and after reaching a minimum value $\hat{K}(z)$ then increases slowly in the transition region which lies between the viscous and log regions of the flow. No attempt to model this physically important transition region has been made in our first order theory, but a next approximation should be sought in which the above mentioned slow variation of $\hat{K}(z)$ is incorporated. Such a programme would involve a relaxation of our K = constant constraint for $z \geqslant z_l$, together with the introduction of a sufficient amount of physical information which will allow a closure (i.e. a finite value of $\max K$). We may therefore regard the present theory as one in which the slowly varying terms $(\zeta\hat{K}'(\zeta)/\hat{K}(\zeta))$ are neglected, and in which the computed $\max K$ is not the final upper bound for $\hat{K}(\infty)$. Our present $\max K$ is identified with some weighted average $\hat{K}(z)$ in the next approximation, this average being taken over the transition and log layers and being somewhat less than the asymptotic value of $\hat{K}(z)$. If this point of view be adopted then it is quite reasonable to test our second inequality by comparing (3.15) with the Laufer and Kline *et al.* measurements, rather than with the bulk discharge determination of von Kármán's constant. The fact that the latter quantity may exceed our present value of $\max K$ by a small amount does not, therefore, rule decisively against our second inequality, the present determination of $\max K$ being regarded only as a first approximation to the asymptotic $\hat{K}(z)$.

There is a second way of modifying the foregoing theory so that it is in clear agreement with observations. We have shown that the first inequality (2.3) is in agreement with observations when *two* dimensional test perturbations $\mathbf{V}_c(x, z)$ are used. Equation (2.3) is therefore also correct for the wider class of all *three* dimensional test perturbations $\mathbf{V}_c(x, y, z)$. If such a relaxation of the first inequality is also used in the statement of the second inequality (3.8) then a smaller value of the minimum possible boundary layer Reynolds number will occur and, according to (3.11), a larger value of $\max K$. The quantitative effect of such a modification of our theory may be obtained from Busse's (1969) generalization of Orr's calculation for Poiseuille flow. When three dimensional perturbations are admitted, Busse found that the minimum Reynolds number was smaller by a factor of 50/88 than the value given by (A 3) (and the minimizing $\mathbf{V}_c$ consists of rolls whose axis are parallel to the flow). From (3.11) we see that the modification in question will increase our $\max K$ by a factor of $(88/50)^{\frac{1}{2}}$ (approximately), thereby giving a $\max K \simeq 0{\cdot}49$ which is definitely larger than the observed values.† There are, moreover, other attractive features in such a modification of our theory, not the least of which is the possibility of incorporating more of the physics of the three dimensional turbulence as constraints on the $\mathbf{V}_c(x, y, z)$. We conclude that our second inequality is, or can readily be modified to be, in agreement with observations and that $\max K$ is close to the observed value.

Since the $U(z)$ profile having largest K has the smallest discharge (for a given τ), this conclusion lends support to Malkus' hypothesis (see § 1) that the realized flow has the least discharge, subject to certain constraints.

5. The inequalities applied to the thermal convection problem

Can the same procedure, with only minor and obvious changes in the formalism, be applied to other one-dimensional turbulent problems, such as the problem of turbulent thermal convection between two differentially heated horizontal plates (see (1.6) for the notation and references)? Figure 2 (*b*) is a sketch of the horizontally averaged temperature gradient $\overline{T}'(z)$ near the lower boundary $z = 0$ when the Nusselt number $D(k^2\nu/g\alpha F)^{-\frac{1}{4}}$ is very large, where

$$F = -k\overline{T}'(0) \tag{5.1}$$

is the total vertical flux of heat between the two boundaries having a temperature difference $2\Delta T_{\frac{1}{2}}$.

For any profile $\overline{T}'(z)$ the thermal boundary layer depth $z_b(\overline{T}')$ and the 'comparison'

† A referee of an earlier draft has remarked that our *first* inequality (2.3) is equally correct if the left-hand side be adjusted downward by changing the number 'one' to (say) 5/11. But this modification would reduce (3.16) by a factor of $(5/11)^{\frac{1}{2}}$ and thereby bring the *second* inequality into even greater conflict with observations. Moreover, this kind of *parametric* adjustment (as contrasted with the conceptual adjustments suggested above) seems quite inappropriate, since the prediction of K provides the only quantitative test of the present theory. The introduction of empirical coefficients would obliterate any heuristic value of the inequalities, such as is illustrated in § 5.

profile $Fk^{-1}(1-z/z_b)$ in figure 2(*b*) are constructed in strict analogy with the corresponding shear profiles in figure 2(*a*), and thus we have

$$-\overline{T}'(z) \geqslant \begin{cases} Fk^{-1}(1-z/z_b), & z < z_b, \\ 0, & z > z_b. \end{cases} \tag{5.2}$$

The observed values of $\overline{T}'_0, z_{b0}$ are connected by a similar inequality.

The assumption pertaining to the energetics of the (thermal) boundary layer states that there exists a two dimensional perturbation in $0 < z < z_{b0}$ which releases as much 'energy' from the mean (temperature) field as that perturbation dissipates, and the precise meaning of this is as follows. In (1.6) $\frac{1}{2}D$ is replaced by z_{b0}, $\mathbf{V}_0$ by $\mathbf{V}_c(x,z)$, and $\mathscr{T}_0$ by the thermal test perturbation $\mathscr{T}_c(x,z)$. The two resulting integrals then provide a functional inequality for the observed $\overline{T}'_0(z)$, in which $\mathbf{V}_c, \mathscr{T}_c$ serve as 'dummy variables'. When the subscripts 'zero' are discarded the two integrals:

$$\frac{k\displaystyle\int_0^{z_b}\overline{(\nabla\mathscr{T}_c)^2}\,dz}{\displaystyle\int_0^{z_b}\overline{w_c\mathscr{T}_c}(-\overline{T}'(z))\,dz} = 1 = \frac{\displaystyle\int_0^{z_b}\nu\overline{(\nabla\times\mathbf{V}_c)^2}\,dz}{\displaystyle\int_0^{z_b}g\alpha\overline{w_c\mathscr{T}c}\,dz} \tag{5.3}$$

together with the boundary conditions stated below, define a manifold M_1 of profiles $\overline{T}'(z)$ to which $\overline{T}'_0(z)$ belongs.

The no-slip boundary condition $w_c = 0 = u_c$ is applied at $z = 0$, and at $z = z_b$ we take w_c to be a maximum as in §3. The isothermal boundary condition $\mathscr{T}_c = 0$ is applied at $z = 0$, and at $z = z_b$ we take $\mathscr{T}_c$ to be a maximum $(\partial\mathscr{T}_c/\partial z = 0)$. In §3 we used $\overline{w_c u_c} < 0$ as a side condition to simplify the formal development (but in retrospect this restriction can be removed without changing the results). Likewise, we will now assume that $\overline{w_c\mathscr{T}_c}$ increases monotonically from zero to z_b, in order to simplify the development of the following inequalities.

When (5.3) is made non-dimensional by using

$$\left.\begin{aligned} \zeta = z/z_b, \quad \xi = x/z_b, \quad \theta = \mathscr{T}/\Delta T_{\frac{1}{2}}, \quad \phi = \frac{\nu\psi}{g\alpha\Delta T_{\frac{1}{2}}z_b^3}, \\ s(\zeta) = -\frac{k}{F}\,\overline{T}'(z), \\ w_c = \partial\psi/\partial x \quad \text{and} \quad u_c = -\partial\psi/\partial z, \end{aligned}\right\} \tag{5.4}$$

and when the non-dimensional version of the boundary conditions are introduced we obtain

$$\frac{z_b^4 g\alpha F}{k^2\nu} = \frac{\displaystyle\int_0^1\overline{(\nabla\theta)^2}\,d\zeta}{\displaystyle\int_0^1\overline{\phi_\xi\theta}s(\zeta)\,d\zeta}, \tag{5.5}$$

$$\int_0^1\overline{(\nabla^2\phi)^2}\,d\zeta = \int_0^1\overline{\phi_\xi\theta}\,d\zeta, \tag{5.6}$$

$$0 = \phi(\xi,0) = \phi_\zeta(\xi,0) = \theta(\xi,0),$$

$$0 = \phi_\zeta(\xi,1) = \theta_\zeta(\xi,1),$$

$$\overline{\phi_\xi\theta} \quad \text{increases monotonically in } 0 < \zeta < 1. \tag{5.7}$$

The non-dimensional temperature gradient (5.4) satisfies $s(0) = 1$, and (5.2) gives $s(\zeta) \geqslant 1-\zeta$. If $\zeta_1 = z_1/z_b$ denotes the non-dimensional tangent point (figure 2*b*) then $s(\zeta_1) = 1-\zeta_1$, and $s'(\zeta_1) = -1$. The counterpart of (1.4) is the assumption that $s(\zeta)$ decreases monotonically† from $z = 0$ to $z = D/2$. These constraints on s are summarized by

$$1-\zeta \leqslant s(\zeta) \leqslant 1, \tag{5.8}$$

$$s(\zeta_1) = 1-\zeta_1, \quad s'(\zeta_1) = -1, \tag{5.9}$$

and

$$s'(\zeta) \leqslant 0. \tag{5.9a}$$

In order to verify the fundamental assumption that (5.3) can be satisfied (when $\overline{T}' = \overline{T}'_0$ and $z_b = z_{b0}$) it is sufficient to show that the observed value of $z_{b0}^4 g\alpha F/k^2\nu$ exceeds the right-hand side of (5.5) [for some permissable (ϕ, θ)] when $s(\zeta)$ is replaced by $1-\zeta \leqslant s_0(\zeta)$. This verification appears in §6.

Equations (5.5)–(5.9) complete the description of the manifold M_1 of $\overline{T}'(z)$, whose lower envelope will be denoted by $\overline{T}'_{e1}(z)$. For any member, the surface temperature $\Delta T_{\frac{1}{2}}$ relative to the temperature at $z = D/2$, is given by

$$\Delta T_{\frac{1}{2}} = -\int_0^{\frac{1}{2}D} \overline{T}'(z)\,dz = \frac{Fz_b}{k}\int_0^{D/2z_b} s(\zeta)\,d\zeta \simeq \frac{Fz_b}{k}\int_0^{\infty} s(\zeta)\,d\zeta, \tag{5.10}$$

where the large Nusselt number approximation has been made in the last term. A lower-bound on the realized surface temperature in M_1 is now computed by minimizing (5.10).

The mathematics necessary for this calculation is partially supplied by the extensive literature on the Rayleigh stability problem for a uniform temperature gradient between *two* rigid boundaries at $\zeta = 0$ and $\zeta = 2$, with $\zeta = 1$ being the plane of symmetry for the temperature θ and streamline ϕ eigenfunctions. Thus we know that the functions which

$$\text{minimize} \quad \frac{\int_0^2 \overline{(\nabla\theta)^2}\,d\zeta}{\int_0^2 \overline{\phi_\xi\theta}\,d\zeta} \quad \text{with} \quad \int_0^2 \overline{(\nabla^2\phi)^2}\,d\zeta = \int_0^2 \overline{\phi_\xi\theta}\,dz\zeta$$

are the rigid–rigid Rayleigh eigenfunctions: $\phi = \phi_R, \theta = \theta_R$. The minimum critical Rayleigh number equals 1708, based on the separation of the two rigid boundaries, and equals $1708/2^4$ if the distance to the midplane $\zeta = 1$ is used as the length scale. Thus we have

$$\min \frac{\int_0^1 \overline{(\nabla\theta)^2}\,d\zeta}{\int_0^1 \overline{\phi_\xi\theta}\,d\zeta} = \frac{1708}{2^4}. \tag{5.11}$$

In view of the symmetry of ϕ_R, θ_R this minimum also applies for the case of our boundary conditions (5.7) at $\zeta = 1$, and the corresponding optimizing (ϕ, θ) are therefore given by the 'bottom half' of the respective Rayleigh eigenfunctions. These have

† This is not necessarily the case at *moderate* Nusselt number where inversions have been reported in the literature.

been tabulated in Chandrasekhar (1961), and have been used in all the numerical calculations cited below.

We will also make use of the following lemma, the proof of which appears in appendix D. If $\overline{\phi_\xi \theta}$ increases monotonically with ζ, and if $s(\zeta)$ decreases monotonically [cf. (5.9a) and the last equation in (5.7)] then

$$\frac{\int_0^1 \overline{\phi_\xi \theta} s(\zeta)\, d\zeta}{\int_0^1 \overline{\phi_\xi \theta}\, d\zeta} \leqslant \int_0^1 s(\zeta)\, d\zeta. \tag{5.12}$$

Since $s(\zeta) \leqslant 1$ a lower bound on (5.5) is

$$\frac{z_b^4 g\alpha F}{k^2 \nu} > \min \frac{\int_0^1 \overline{(\nabla\theta)^2}\, d\zeta}{\int_0^1 \overline{\phi_\xi \theta}\, d\zeta} = \frac{1708}{2^4},$$

and when this bound on z_b is inserted into (5.2), we obtain a broken straight line which is a lower bound for all the $-\overline{T}'(z)$ profiles including $\overline{T}'_0(z)$. Furthermore, the elimination of z_b from (5.5) and (5.10) gives

$$\Delta T_{\frac{1}{2}} = \frac{F}{k}\left(\frac{k^2\nu}{g\alpha F}\right)^{\frac{1}{4}} \int_0^\infty s(\zeta)\, d\zeta \left[\frac{\int_0^1 \overline{\phi_\xi \theta}\, d\zeta}{\int_0^1 \overline{\phi_\xi \theta s}\, d\zeta}\right]^{\frac{1}{4}} \left[\frac{\int_0^1 \overline{(\nabla\theta)^2}\, d\zeta}{\int_0^1 \overline{\phi_\xi \theta}\, d\zeta}\right]^{\frac{1}{4}}, \tag{5.13}$$

$$\left.\begin{aligned} \Delta T_{\frac{1}{2}} &\geqslant \frac{F}{k}\left(\frac{k^2\nu}{g\alpha F}\right)^{\frac{1}{4}} \int_0^\infty s(\zeta)\, d\zeta \left[\int_0^1 s(\zeta)\, d\zeta\right]^{-\frac{1}{4}} \left(\frac{1708}{2^4}\right)^{\frac{1}{4}} \\ &\geqslant \frac{F}{k}\left(\frac{k^2\nu}{g\alpha F}\right)^{\frac{1}{4}} \left[\int_0^1 (1-\zeta)\, d\zeta\right]^{\frac{3}{4}} \left(\frac{1708}{2^4}\right)^{\frac{1}{4}}, \end{aligned}\right\} \tag{5.14}$$

where $s(\zeta) \geqslant 0$, $s(\zeta) \geqslant (1-\zeta)$ have been used in the last line, and (5.11) and (5.12) have been used in the preceding line. This bound on the surface temperature can be written as

$$C \equiv \frac{g\alpha F(\nu/k^2)^{\frac{1}{3}}}{(2g\alpha \Delta T_{\frac{1}{2}})^{\frac{4}{3}}} < 0{\cdot}167, \tag{5.15}$$

where C is a conventionally measured non-dimensional heat flux.

We now proceed to the second (or 'similarity') inequality, which is introduced in strict correspondence with the shear flow problem (see § 3).

The 'law of the wall' for the thermal problem asserts that $\overline{T}_0(z)$ approaches a function $\overline{T}_{0w}(z)$ which is independent of $D \to \infty$, provided z and F are held constant. This asymptotic $\overline{T}_{0w}$ is then assumed to approach zero with increasing z according to the power law $z^{-\beta}$, where the exponent β is undetermined. A bound on β arises from the following argument which shows that the internal energy function

$$\Omega_0(z) \equiv \int_0^z \overline{T}_{0w}\, dz \tag{5.16}$$

must *diverge* as $z \to \infty$.

Let us view the limiting case $D = \infty$ of the parallel plate convection problem from the point of view of an initial value problem, in which a steady heat flux F is suddenly applied at time $t = 0$ to the lower boundary of a semi-infinite fluid having $\overline{T}_0(z,t) = 0$ as its initial temperature. For $t > 0$ the conservation of energy requires

$$\int_0^\infty \overline{T}_0(z,t)\,dz = Ft. \tag{5.17}$$

We assume that as time increases a statistically steady $\overline{T}_0(z,t) = \overline{T}_{0w}(z)$ is established at all *finite* z, with the internal energy (5.17) appearing at successively greater heights as t increases. Consistency between (5.16) and (5.17) then requires that Ω_0 must increase with increasing z.

It then follows that the exponent in $\overline{T}_{0w}\alpha z^{-\beta}$ is $\beta \leqslant 1$. Therefore $-zT'_{0w}/T_{0w} \leqslant 1$ for large z, and the asymptote to Ω_0 may be bounded by a logarithmic curve. In order to enforce this bound we shall constrain the profiles of

$$\Omega(z) \equiv \int_0^z \overline{T}(z)\,dz, \tag{5.18}$$

to satisfy (5.5)–(5.9a) and

$$\frac{z\Omega''(z)}{\Omega'(z)} = -1, \quad \text{for} \quad z > z_l; \tag{5.19}$$

$$z_l > z_b(\overline{T}'), \quad \text{or} \quad z_b \geqslant z_l > z_1, \quad \text{or} \quad z_1(\overline{T}') \geqslant z_l > 0; \tag{5.20}$$

where z_l denotes the height at which the logarithmic region for Ω starts. The physical content of this procedure appears with the assumption that the envelope Ω'_{e2}, for *each* of the three inequalities in (5.20), bounds Ω'_0 from below.

It is readily shown, as was the case in §3, that the first two inequalities in (5.20) have envelopes which are identical to the envelope for M_1, and in these cases the above mentioned assumption is trivially correct. The only new information in (5.19) and (5.20) is supplied by the last inequality, which then gives us the manifold

$$M_2: \begin{cases} M_1 \quad \text{and} \\ \dfrac{z\Omega''}{\Omega'} = -1, \quad z > z_l, \end{cases} \tag{5.21}$$

$$z_l \leqslant z_1(\overline{T}'); \tag{5.22}$$

and the statement of the physical assumption is

$$\Omega'_{e2}(z) \leqslant \Omega'_0(z). \tag{5.22a}$$

In order to determine the envelope Ω'_{e2} we first write the solution of the differential equation (5.21) in the form

$$\Omega(z) = \left(\frac{F}{k}\right)\left(\frac{k^2\nu}{g\alpha F}\right)^{\frac{1}{2}} K_T^{-1} \ln z \left(\frac{k^2\nu}{g\alpha F}\right)^{-\frac{1}{2}} + \text{constant}, \tag{5.23}$$

where K_T is a non-dimensional constant of integration. The corresponding temperature field is the derivative of this, and the temperature gradient,

$$\frac{\overline{T}'(z)}{F/k} = -\frac{1}{z^2 K_T}\left(\frac{k^2\nu}{g\alpha F}\right)^{\frac{1}{2}}, \tag{5.24}$$

can be written in the non-dimensional form

$$\zeta^2 s(\zeta) = \frac{1}{K_T z_b^2}\left(\frac{k^2\nu}{g\alpha F}\right)^{\frac{1}{2}}. \tag{5.25}$$

According to (5.22) this relation and its derivative are valid at

$$\zeta_1 = z_1/z_b$$

at which point we also have $s(\zeta_1) = 1-\zeta_1$, and $s'(\zeta_1) = -1$. By substituting the latter into (5.25), or $s'(\zeta)/s = -2/\zeta$, we obtain $\zeta_1 = \frac{2}{3}$, $s(\zeta_1) = \frac{1}{3}$, and

$$s(\zeta) = \frac{4}{27\zeta^2} \quad \text{for} \quad \zeta \geqslant \tfrac{2}{3}. \tag{5.26}$$

Equations (5.25)–(5.26) then give

$$z_b^2 = \frac{27}{4K_T}\left(\frac{k^2\nu}{g\alpha F}\right)^{\frac{1}{2}}, \tag{5.27}$$

and when z_b is eliminated by means of (5.5) we obtain

$$K_T = \frac{27}{4}\left[\frac{\int_0^1 \overline{\phi_\xi\theta}s(\zeta)\,d\zeta}{\int_0^1 \overline{\phi_\xi\theta}\,d\zeta}\right]^{\frac{1}{2}}\left[\frac{\int_0^1 \overline{\phi_\xi\theta}\,d\zeta}{\int_0^1 \overline{(\nabla\theta)^2}\,d\zeta}\right]^{\frac{1}{2}}. \tag{5.28}$$

The maximum value of this with respect to s, θ, ϕ determines the minimum value of Ω' at large z, and also determines the asymptotic slope of the envelope $\Omega_{e2}(z)$ of the manifold.

To compute the maximum value of K_T we note that

$$s(\zeta) \leqslant s_{\max} = \begin{cases} 1, & \zeta < \frac{2}{3} \\ \dfrac{4}{27\zeta^2}, & \zeta > \frac{2}{3} \end{cases} \tag{5.29}$$

and therefore (5.28) becomes

$$\max_{(s,\theta,\phi)} K_T = \frac{27}{4}\max_{(\theta,\phi)}\left[\frac{\int_0^1 \overline{\phi_\xi\theta}s_{\max}\,d\zeta}{\int_0^1 \overline{\phi_\xi\theta}\,d\zeta}\right]^{\frac{1}{2}}\left[\frac{\int_0^1 \overline{\phi_\xi\theta}\,d\zeta}{\int_0^1 \overline{(\nabla\theta)^2}\,d\zeta}\right]^{\frac{1}{2}}. \tag{5.30}$$

From (5.11) and (5.12) it follows that

$$\max K_T < \frac{27}{4}\left[\int_0^1 s_{\max}\,d\zeta\right]^{\frac{1}{2}}\left[\frac{2^4}{1708}\right]^{\frac{1}{2}} = 0{\cdot}56,$$

where (ϕ_R, θ_R) supply the optimizing eigenfunctions for this rigorous, but crude, upper bound on K_T. A much sharper value can be obtained by noting that the middle term (containing $s_{\max}$) in (5.30) is rather insensitive to permitted variations in $\overline{\phi_\xi\theta}$, and therefore (ϕ_R, θ_R) may be used in evaluating the middle term. The following term in (5.30) is, of course, maximized by (ϕ_R, θ_R). Thus a 'good' approximation to (5.30) is obtained by merely substituting the Rayleigh eigenfunctions, and the numerical result is

$$\max K_T \simeq \frac{27}{4}(0{\cdot}45)^{\frac{1}{2}}\left(\frac{2^4}{1708}\right)^{\frac{1}{2}} = 0{\cdot}44. \tag{5.31}$$

The 'optimal' $\overline{T}$ and the optimal internal energy function (5.18) which is associated with this K_T is computed as follows. The boundary layer width (5.27) is given by

$$z_b\left(\frac{k^2\nu}{g\alpha F}\right)^{-\frac{1}{4}} = \left(\frac{27}{4K_T}\right)^{\frac{1}{2}} = 3{\cdot}9 \tag{5.31a}$$

and the height $z = 2z_b/3$ at which the 'log layer' begins is given by

$$z_1\left(\frac{k^2\nu}{g\alpha F}\right)^{-\frac{1}{4}} = 2{\cdot}6. \tag{5.31b}$$

When (5.29) is used in (5.10) the surface temperature becomes

$$\Delta T_{\frac{1}{2}} = \tfrac{8}{9}Fz_b/k \tag{5.31c}$$

and the use of (5.31a) then gives

$$g\alpha\Delta T_{\frac{1}{2}}/[g\alpha F^3k^{-1}]^{\frac{1}{4}} = 3{\cdot}47(\nu/k)^{\frac{1}{4}}. \tag{5.31d}$$

The 'optimal' temperature field then decreases linearly until $z = z_1$, is reached, at which height $\overline{T}(z)$ is given by

$$\overline{T}(z_1)/\Delta T_{\frac{1}{2}} = 1 - Fz_1/k\Delta T_{\frac{1}{2}} = 1 - \tfrac{2}{3}\times\tfrac{9}{8} = \tfrac{1}{4}. \tag{5.31e}$$

At this and larger z the temperature follows the z^{-1} law:

$$\overline{T}(z) = \frac{F}{k}\left(\frac{1}{zK_T}\right)\left(\frac{k^2\nu}{g\alpha F}\right)^{\frac{1}{2}}. \tag{5.31f}$$

In order to compare the foregoing profile with subsequently cited experiments in air ($\nu/k = 0{\cdot}72$) we will write the above results in terms of the new non-dimensional variables

$$z_* = z(k^3/g\alpha F)^{-\frac{1}{4}} \tag{5.32a}$$

$$T_*(z_*) = g\alpha\overline{T}(z)/[(g\alpha F)^3k^{-1}]^{\frac{1}{4}} \tag{5.32b}$$

in which case the 'optimal' internal energy field may be expressed as:

$$\left.\int_0^{z_*} T_*(x)\,dx = \begin{cases} 3{\cdot}2z_* - z_*^2/2, & z_* < 2{\cdot}4 \\ 1{\cdot}92\ln(z_*/2{\cdot}4) + 4{\cdot}8, & z_* > 2{\cdot}4 \end{cases} \quad (\nu/k = 0{\cdot}72).\right\} \tag{5.33}$$

For large z_* our theory requires the observed internal energy function to be larger than (5·33).

At small values of z the envelope is less than that which is obtained from (5.33), and the minimum possible surface temperature is smaller than (5.31d). To obtain $\min\Delta T_{\frac{1}{2}}$ from (5.13) we first note that

$$s(\zeta) \geqslant s_{\min} = \begin{cases} 1-\zeta, & \zeta < \frac{2}{3}, \\ 4/27\zeta^2, & \zeta > \frac{2}{3}, \end{cases}$$

in consequence of which we have

$$\frac{\int_0^\infty s(\zeta)\,d\zeta}{\left[\int_0^1 s(\zeta)\,d\zeta\right]^{\frac{1}{4}}} = \frac{\int_0^1 s(\zeta)\,d\zeta + \frac{4}{27}}{\left[\int_0^1 s(\zeta)\,d\zeta\right]^{\frac{1}{4}}} > \frac{\int_0^1 s_{\min}\,d\zeta + \frac{4}{27}}{\left[\int_0^1 s_{\min}\,d\zeta\right]^{\frac{1}{4}}} = \frac{18}{27}\times\left(\frac{27}{14}\right)^{\frac{1}{4}}.$$

When this inequality is used in (5.14) we get

$$\frac{[(g\alpha F)^3 \nu/k^2]^{\frac{1}{4}}}{g\alpha \Delta T_{\frac{1}{2}}} > 2{\cdot}53, \tag{5.34}$$

and in terms of the C defined by (5.15) this bound is equivalent to

$$C < 0{\cdot}115. \tag{5.35}$$

We are now in a position to verify the main assumption (5.22a).

6. Comparison with experiment and summary

The temperature in the air above a heated horizontal plate has been measured by Townsend (1959). The abscissa of his figure 2 is (5.32a) and the ordinate reduces to (5.32b) if one makes a Boussinesq extrapolation of the experimental results. By smoothing and differentiating the experimental T_* data we obtained a curve like our figure 2b, and by drawing the tangent line the boundary layer thickness was found to be (slightly) larger than $z_{b0}(g\alpha F/k^2\nu)^{\frac{1}{4}} = 5{\cdot}4$. A similar value of

$$z_{b0}(g\alpha F/k^2\nu)^{\frac{1}{4}} = 5{\cdot}3$$

was obtained from the measurements of Deardorff & Willis (1967), even though the Nusselt number in this experiment was only thirty. Our inequality [cf. (5.3)] pertaining to the energetics of the thermal boundary layer can be immediately verified by showing that these numbers, or $z_{b0}^4(g\alpha F/k^2\nu) \simeq (5{\cdot}3)^4$ exceed the value of the right-hand side of (5.5) for some permissible (ϕ, θ, s). Accordingly, we substituted $s = 1-\zeta$ and the Rayleigh functions ϕ_R, θ_R in (5.5), obtaining a value $4{\cdot}5^4$ which is definitely less than the experimental values. Thus we conclude that the inequality is correct for the thermal problem [as well as being correct for the shear flow problem (see §4)]. This inequality for the thermal problem implies that (5.15) must be a bound for the heat flux.

The main quantitative result for the thermal problem is the upper bound $K_T = 0{\cdot}44$ (5.31) for the z^{-1} law (5.31f). When air is the working medium, and when (5.22a, b) are used, the theoretical z^{-1} law becomes $T_*(z_*) = 1{\cdot}9z_*^{-1}$. This is a fair lower bound for Townsends experiment since the best fit is $T_* = 2{\cdot}6z_*^{-1}+0{\cdot}06$ for $z_* > 8$, and since $T_* = 2{\cdot}0z_*^{-1}+0{\cdot}07$ is a possible fit for $z_* > 10$. However, Deardorff & Willis (1967) have questioned the validity of the z^{-1} law.

We also note that (5.31d) is *close* to the ordinate at $z_* = 0$ on Townsend's measured temperature profile, and that our bound (5.34) on the surface temperature is definitely *less* than the measured value. We therefore conclude that the key inequality (5.22a) is correct for small z as well as for large z. For the intermediate values of z we have integrated Townsend's T_*, since this eliminates some of the scatter, and compared the result with (5.33). The observed Ω_0 agrees with (5.33) up to $z_* = 1{\cdot}5$, at which point Ω_0 begins to *exceed* our 'optimal' Ω. We therefore conclude that the similarity inequality is correct for all values of z in the wall region.

It appears from the previous paragraph that the maximum value of K_T is close to Townsend's observed value, and the maximum value of von Kármán's constant is also close to the observed value (see §4). Although the reason for this close agreement is unknown, it does suggest that some general optimal principle is applicable to the fully turbulent regimes. We also lack a theoretical explanation for our inequality pertaining to the energetics of the boundary layer, and the value of the foregoing work would be greatly enhanced if a more satisfactory physical basis for the inequalities could be found. The question also arises as to whether there are additional inequalities which could be incorporated, so as to give a better lower bound for the *entire* profiles.

In the absence of answers to these questions, our formalism might be tested and generalized by applying it to other 'one-dimensional' turbulence problems, such as 'double-diffusive convection' (Linden & Shirtcliffe 1978), between two very deep fluids containing different concentrations of two solutes. The problem is to compute the *two* fluxes across the self-adjusting interface which separates the two deep layers.

I would like to thank Prof. Klaus Hasselmann for inviting me to the Max-Planck-Institut für Meteorologie. Part of this work was carried out at that institution and also at the Geophysical Fluid Dynamics Summer Program under contract with the Office of Naval Research. Thanks are also due to Dr Colin Shen for the numerical calculation of the Orr Eigenfunctions.

Appendix A. The Orr variational problem and verification of (2.3)

We must show that the right-hand side of (2.3) can be made less than unity for some permissible $\mathbf{V}_c$, where the boundary layer depth z_{b0} is obtained from the *observed* U_0', by means of the tangent line construction in figure 2 (*b*). Such a construction was made on the experimental curve in figure 26 of Laufer (1954), which is a plot of directly measured shear (and Reynolds stress) in the neighbourhood of the wall of a pipe for which the overall Reynolds number was 500000. The boundary layer depth so obtained is given by

$$\frac{\nu}{\tau^{\frac{1}{2}} z_{b0}} = \frac{1}{22}. \tag{A 1}$$

When (2.1) is substituted into (2.3) and when the non-dimensionalization in (2.8) is used we obtain the expression

$$\frac{\nu^2}{\tau z_{b0}^2}\left[\frac{\int_0^1 d\zeta\overline{(\nabla^2\psi)^2}+\int_1^{D/2z_{b0}} d\zeta\overline{(\nabla^2\psi)^2}}{\int_0^1 \overline{\psi_\xi\psi_\zeta}(1-\zeta)\,d\zeta}\right] \tag{A 2}$$

for the right-hand side of (2.3), and thus we must show that the term in brackets can be made less than $(22)^2$ by using some permissible ψ. For this purpose it is necessary to discuss the eigenfunctions which occur in Orr's investigation of the 'absolute' stability of laminar Poiseuille flow.

For a parabolic flow having maximum shear B in a channel of height $2a$, Orr (also see Lin 1955) found a minimum Reynolds number $Ba^2/\nu = 2\times 88$. The optimizing stream-function is symmetric about the channel centre, in consequence of which we may express the energy integrals in terms of the half range $\frac{1}{2}D$ [cf. (1.5)], and thus Orr's result can be written as

$$\min \frac{\int_0^1 \overline{(\nabla^2\psi)^2}\, d\zeta}{\int_0^1 \overline{\psi_\xi \psi_\zeta}(1-\zeta)\, d\zeta} = 2\times 88 \tag{A 3}$$

$$\psi(\xi,0) = \psi_\zeta(\xi,0) = 0;\quad \psi \text{ is symmetric about } \zeta = 1 \tag{A 4}$$

$$\text{optimal } \psi = \psi_{\text{orr}} = Re\, Y_0(\zeta)\exp{(il\xi)}. \tag{A 5}$$

Since the lower half $(0<\zeta<1)$ of Orr's eigenfunction in a channel satisfies our boundary conditions (2.12) and (2.13), it is permissible to use this eigenfunction in (A 2) for $\zeta \geqslant 1$. For $\zeta \leqslant 1$ we will choose a function which satisfies $\nabla^4\psi = 0$, which vanishes as $\zeta\to\infty$, and which is such that ψ_ζ is continuous at $\zeta = 1$. The biharmonic solution for $\zeta \geqslant 1$ is readily found to be

$$\psi = Re\, Y_0(1)\exp{(il\xi)}\{\exp[-l(\zeta-1)] + l(\zeta-1)\exp{[-l(\zeta-1)]}\}. \tag{A 6}$$

For $D/z_{b0}\to\infty$ the bracketed term in (A 2) then simplifies to

$$\frac{\int_0^1 d\zeta\,|Y_0''(\zeta) - l^2 Y_0| + 2l|Y_0(1)|^2}{l\int_0^1 d\zeta(1-\zeta)\,\text{Im}\, Y_0^*\, Y_0'(\zeta)}, \tag{A 7}$$

where Y_0^* is the complex conjugate of Y_0 and Im denotes imaginary part. We note that if the second term in the numerator is ignored then (A 7) is the same as (A 3), the value of which is definitely less than 22^2. Therefore our inequality will be established if we can show that the contribution of the ignored term $2l|Y_0(1)|^2$ is sufficiently small.

Since Orr did not explicitly evaluate Y_0, we have performed the calculation [correcting an obvious typographical error in one of the signs in Orr's (67)], using 18 terms in the power series expansion for ψ and Orr's value of $l = (4{\cdot}4)^{\frac{1}{2}}$. The eigenfunction and its derivatives are plotted in figures 4(a, b); with the values of Y'' near the endpoints (0, 1) being excluded because they are not (computationally) reliable. When our approximation to ψ_{Orr} was substituted in (A 3) the result obtained was 186, with the error $186-2(88)$ being attributed to the endpoint error. From these eigenfunctions we obtained $2l|\,Y_0^2(1)| = 2(4{\cdot}4)^{\frac{1}{2}}$, as compared with a value of 75 for the first term in the numerator of (A 7). Therefore the expression in (A 7) equals 187 and we conclude that (A 2) is definitely less than unity for some ψ. This completes the verification of the main assumption (2.3).

It is necessary to consider some further aspects of the minimum value of the functional

$$J = \frac{\int_0^1 \overline{(\nabla^2\psi)^2}\, d\zeta}{\int_0^1 \overline{\psi_\xi\psi_\zeta}\, s(\zeta)\, d\zeta} \tag{A 8}$$

for various values of $s(\zeta)$. For $s = 1-\zeta$ Orr's work assures us that (A 3) is the minimum value of (A 8) for symmetric functions (which satisfy a *no slip* boundary condition at $\zeta = 2$), and we will show that our no stress boundary condition (2.13) at $\zeta = 1$ gives the same minimum value for (A 8).

For any s, the function which minimizes (A 8) satisfies the Euler equation

$$\nabla^4\psi + Js(\zeta)\,\psi_{\xi\xi} + \tfrac{1}{2}Js'(\zeta)\,\psi_\xi = 0 \tag{A 8a}$$

and the contribution of the free endpoint $\zeta = 1$ to the variation of (A 8) is

$$0 = -\overline{\nabla^2\psi(\xi,1)\,\delta\psi_\zeta(\xi,1)} + \overline{\delta\psi\nabla^2\psi_\zeta} + \tfrac{1}{2}Js(1)\,\overline{\psi_\xi\,\delta\psi(\xi,1)}. \tag{A 9}$$

If the stress (2.13) vanishes at $\zeta = 1$ then either the vertical velocity vanishes

$$[\psi(\xi,1) = 0]$$

at $\zeta = 1$ or

$$\frac{\partial\psi(\xi,1)}{\partial\zeta} = 0 \tag{A 10}$$

and the latter condition will lead to a smaller minimum value of J than the former [see p. 131 of Orr (1907)]. With the boundary condition (A 10) the first term on the right of (A 9) vanishes. The last term in (A 9) vanishes for the case $s = 1-\zeta$, and (A 9) then reduces to $0 = \overline{\delta\psi(\xi,1)\,\psi_{\zeta\zeta\zeta}(\xi,1)}$. Since $\psi(\xi,1) \neq 0$, the *single* condition (A 10) implies that

$$\frac{\partial^3\psi(\xi,1)}{\partial\zeta^3} = 0 \tag{A 11}$$

is the second boundary condition for (A 8) to have a minimum when $s = 1-\zeta$. Now we note that (A 10) and (A 11) are also satisfied by Orr's *symmetric* eigenfunction for the interval $0 \leqslant \zeta \leqslant 2$, and thus we conclude that the minimum J for our problem is the same as the minimum in Orr's problem, i.e.

$$\min \frac{\int_0^1 \overline{(\nabla^2\psi)^2}\,d\zeta}{\int_0^1 \overline{\psi_\xi\psi_\zeta}(1-\zeta)\,d\zeta} = 2\times 88, \tag{A 12}$$

$$\psi(\xi,0) = 0 = \psi_\zeta(\xi,0),$$

$$\psi_\zeta(\xi,1) = 0 \quad \text{(only)}.$$

Orr also found a minimum Reynolds number of 44·3 (based on the shear and the half channel width) for the absolute stability of laminar *Couette* flow. This corresponds to $s = 1$ with rigid boundary conditions at $\zeta = (0,2)$, and thus we have the formal relation

$$\min \frac{\int_0^2 \overline{(\nabla^2\psi)^2}\,d\zeta}{\int_0^2 \overline{\psi_\xi\psi_\zeta}\,d\zeta} = 44{\cdot}3, \tag{A 13}$$

$$0 = \psi(\xi,0) = \psi_\zeta(\xi,0),$$

$$0 = \psi(\xi,2) = \psi_\zeta(\xi,2).$$

The optimizing ψ for this problem has 'mirror symmetry' about the origin

$$(\xi = 0, \zeta = 1),$$

where $\psi(\xi, \zeta)$ has its maximum amplitude. This means that if $\Delta \equiv 1-\zeta$ then

$$\psi(-\xi, -\Delta) = \psi(\xi, \Delta).$$

Moreover the Reynolds stress $\overline{\psi_\xi \psi_\zeta}$ is a *maximum* at $\zeta = 1$, whereas for our boundary condition (2.13) the stress is *zero*. This suggests that the minimum of J for $s = 1$ and for our boundary conditions is greater than Orr's value, or

$$\min \frac{\int_0^1 \overline{(\nabla^2\psi)^2}\, d\zeta}{\int_0^1 \overline{\psi_\xi \psi_\zeta}\, d\zeta} > 44{\cdot}3, \qquad \text{(A 14)}$$

$$0 = \psi(\xi, 0) = \psi_\zeta(\xi, 0),$$

$$\psi_\zeta(\xi, 1) = 0.$$

This may be proven indirectly, by tentatively assuming that the inequality in (A 14) is incorrect, and by then showing that this leads to a contradiction (with Orr's result). Accordingly we assume that the optimal ψ for the left-hand side of (A 14) is such that the value of the latter is *less* than 44·3. Since this ψ is only defined for the interval $\zeta \leqslant 1$, we may reflect ψ about $\xi = 0, \zeta = 1$, so as to obtain a function which is defined for the interval $0 \leqslant \zeta \leqslant 2$. The latter function and its first derivative

$$\psi_\zeta(\xi, 1) = 0$$

are continuous at $\zeta = 1$, but the vorticity has a finite discontinuity. The piecewise evaluation of the integrals on the left side of (A 13) have a ratio which is less than 44·3, according to the tentative assumption made above. This function (with continuous ψ_ζ) can now be smoothed at $\zeta = 1$ in such a way as to eliminate the vorticity discontinuity, and without significant change in ψ or ψ_ζ outside the smoothing region. It follows that the smoothed function still yields a value of

$$\int_0^2 \overline{(\nabla^2\psi)^2}\, d\zeta \Big/ \int_0^2 \overline{\psi_\xi \psi_\zeta}\, d\zeta$$

which is less than 44·3, and this is in contradiction with the minimum (A 13) obtained by Orr. Therefore the tentative assumption made above is false, and the inequality stated in (A 14) is correct.

These results can be used in connexion with the minimum discharge relation (2.15). Since $s \leqslant 1$ the last term in (2.15) is greater than the square root of (A 14), and the preceding term in (2.15) is greater than $\int_0^1 (1-\zeta)\, d\zeta = \frac{1}{2}$, because $s \geqslant 1-\zeta$. Therefore the minimum in (2.15) exists, and $Q_0(\tau^{\frac{1}{2}}D)^{-1} > \frac{1}{2}(44{\cdot}3)^{\frac{1}{2}}$. A better estimate of the minimum drag coefficient can be obtained by using $s = 1-\zeta$ for both terms in (2.15) (but this does not give the exact minimum). We have not pursued this mathematical problem because the observed drag coefficient is not a constant.

Appendix B. Calculation of von Kármán's constant (3.16)

Equation (A 14) implies that (3.15) 'exists', and the numerical value is computed by the following considerations.

Let K_a denote the maximum value of K obtained from (3.12) when

$$s(\zeta) = s_a(\zeta) = 1-\zeta.$$

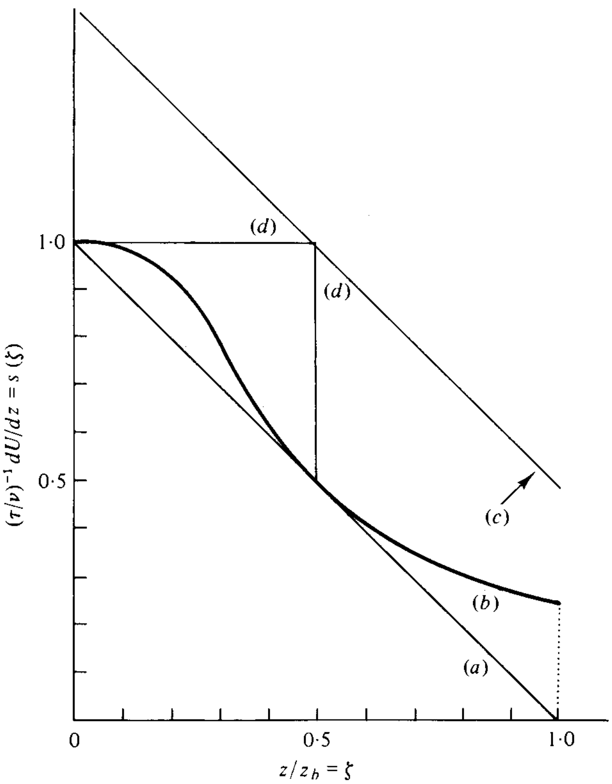

FIGURE 3. The discontinuous curve which is labelled (d) for $\zeta < \frac{1}{2}$ and labelled (b) for $\zeta > \frac{1}{2}$ is the curve $s(\zeta)$ which gives the maximum value of von Kármán's constant in appendix B. The continuous curve labelled (b) and the straight line labelled (c) are used for comparison purposes (see text).

This line is drawn in figure 3 together with the curves s_b, s_c, s_d, the latter being the one which appears in (3.15). From (A 12) we have

$$K_a = \frac{4}{(2 \times 88)^{\frac{1}{2}}} = 0{\cdot}302 \tag{B 1}$$

with ψ_{Orr} being the optimizing function. Obviously $(K_a/4)^2$ must be less than (3.15), and we turn next to the smooth curve denoted by (b) in figure 3. The specific way in which this s_b was constructed is not important for the present discussion, suffice it to say that $s_b'(0) = 0 = s_b''$ and s_b joins smoothly on to $s = 1/4\zeta$ at $\zeta = \frac{1}{2}$. If K_b denotes the value of K obtained by substituting s_b and ψ_{Orr} in (3.12) then the computed integrals give

$$K_b = 0{\cdot}325 \tag{B 2}$$

and it is obvious that this must also be less than (3.15).

Now consider the right-hand side of (3.15), or

$$\left[\frac{\int_0^1 \overline{\psi_\xi \psi_\zeta (1-\zeta)}\, d\zeta}{\int_0^1 \overline{(\nabla^2\psi)^2}\, d\zeta}\right] \frac{\int_0^1 \overline{\psi_\xi \psi_\zeta}\, s_d(\zeta)\, d\zeta}{\int_0^1 \overline{\psi_\xi \psi_\zeta}(1-\zeta)\, d\zeta}.$$

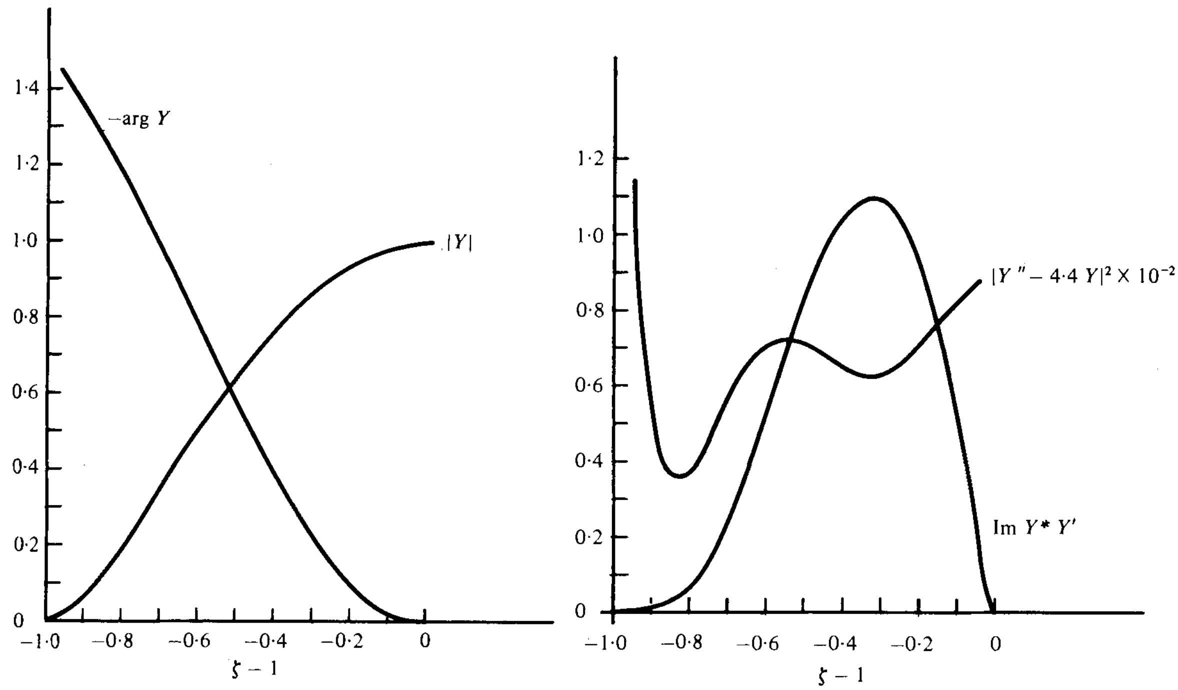

FIGURE 4. (*a*) The amplitude $|Y|$ and the (negative) phase angle (rad) of the Orr eigenfunction for a parabolic velocity profile. (*b*) The corresponding values of the 'Reynolds stress' Im Y^*Y' and the viscous dissipation for the Orr eigenfunctions.

If we assume a small change in the *form* of the optimal $\overline{\psi_\xi \psi_\zeta}$ (cf. figure 4) as s increases from $1-\zeta$ to $s_d(\zeta)$ then the ψ which maximizes the above expression is the same as the ψ which maximizes the bracketed term. Therefore $\psi = \psi_{\text{Orr}}$ should give a good first approximation to (3.15), and when the integration is carried out we obtain

$$K_d = 0{\cdot}362. \tag{B 3}$$

This number is a lower bound to the exact value of (3.15), and we shall now calculate a (crude) upper bound. The curve $s_c = 1-\zeta+\frac{1}{2}$ in figure 3 obviously exceeds the maximum possible $s = s_d$ at all values of ζ and consequently (3.15) gives

$$\max\left(\frac{K}{4}\right)^2 < \max_{\psi} \frac{\int_0^1 \overline{\psi_\xi \psi_\zeta}(1-\zeta+\tfrac{1}{2})\,d\zeta}{\int_0^1 \overline{(\nabla^2\psi)^2}\,d\zeta}$$

$$< \max \frac{\int_0^1 \overline{\psi_\xi \psi_\zeta}(1-\zeta)\,d\zeta}{\int_0^1 \overline{(\nabla^2\psi)^2}\,d\zeta} + \tfrac{1}{2}\max \frac{\int_0^1 \overline{\psi_\xi \psi_\zeta}\,d\zeta}{\int_0^1 \overline{(\nabla^2\psi)^2}\,d\zeta} \leqslant \frac{1}{2\times 88} + \frac{1}{2\times 44{\cdot}3}$$

$$\max K < 0{\cdot}52,$$

where (A 12) and (A 14) have been used. Combining this with (B 3) gives

$$0{\cdot}362 < \max K \leqslant 0{\cdot}52, \tag{B 4}$$

where the second inequality means that the last number is a very crude upper bound and that the true value of max K probably lies very much closer to 0·362 than to 0·52. The author's guess is that $\max K = 0{\cdot}37 \pm 0{\cdot}01$.

Appendix C. The Orr variational problem for the observed turbulent boundary layer

The purpose of this note is to document the assertion made at the end of the second paragraph of § 2, namely, that $U_0(z)$ is unstable (in the sense of Orr) to two-dimensional perturbations when the Reynolds number $\tau^{\frac{1}{2}}D/\nu \to \infty$. The following proof is based on the fact that above a certain $\tau^{\frac{1}{2}}z/\nu \simeq 20$ the observed $dU_0/d\ln z$ decreases monotonically as the asymptotic value is approached at large $\tau^{\frac{1}{2}}z/\nu$. Thus in this range we have

$$\frac{z}{\tau^{\frac{1}{2}}}\frac{dU_0}{dz} \geqslant \frac{1}{K_0} \tag{C 1}$$

where K_0 is the observed value of von Kármán's constant.

Let $\psi(x,z)$ be an arbitrary stream function which satisfies the no-slip conditions at $z = 0$ and which is vanishingly small for $z\tau^{\frac{1}{2}}/\nu \to \infty$. For this test function the ratio of dissipation to energy release is

$$\frac{\nu\int_0^\infty \overline{(\nabla^2\psi)^2}\,dz}{\int_0^\infty \overline{\psi_x\psi_z}\,U_0'(z)\,dz} < \frac{\nu\int_0^{h\nu\tau^{-\frac{1}{2}}} \overline{(\nabla^2\psi)^2}\,dz + \nu\int_{h\nu\tau^{-\frac{1}{2}}}^\infty \overline{(\nabla^2\psi)^2}\,dz}{\int_0^{h\nu\tau^{-\frac{1}{2}}} \overline{\psi_x\psi_z}\,U_0'\,dz}, \tag{C 2}$$

where h is any number greater than 20. The observed U_0 is unstable in the sense of Orr if there exists a ψ which makes the left-hand side of (C 2) less than unity. For $z = h\nu\tau^{\frac{1}{2}}$ the quantity $U_0'(z) \geqslant U_0'(h\nu/\tau^{\frac{1}{2}}) \geqslant \tau/h\nu K_0$ decreases monotonically, and therefore is smaller than

$$\frac{\nu\int_0^{h\nu\tau^{-\frac{1}{2}}} dz\overline{(\nabla^2\psi)^2} + \nu\int_{h\nu\tau^{-\frac{1}{2}}}^\infty dz\overline{(\nabla^2\psi)^2}}{U_0'\left(\frac{\tau^{\frac{1}{2}}}{h\nu}\right)\int_0^{h\nu\tau^{-\frac{1}{2}}} dz\,\overline{\psi_x\psi_z}} < \frac{K_0\nu^2 h}{\tau}\left[\frac{\int_0^{h\nu\tau^{-\frac{1}{2}}} dz\,\overline{(\nabla^2\psi)^2} + \int_{h\nu\tau^{-\frac{1}{2}}}^\infty dz\overline{(\nabla^2\psi)^2}}{\int_0^{h\nu\tau^{-\frac{1}{2}}} dz\,\overline{\psi_x\psi_z}}\right]$$

$$< \frac{K_0}{h}\left[\frac{\int_0^1 \overline{(\nabla^2\psi)^2}\,d\zeta + \int_1^\infty \overline{(\nabla^2\psi)^2}\,d\zeta}{\int_0^1 \overline{\psi_\xi\psi_\zeta}\,d\zeta}\right]. \tag{C 3}$$

Now it is obvious that a permissible $\psi = \operatorname{Re} e^{i\xi}\phi(\zeta)$ can be chosen such that the bracketed term is a positive number. With this ψ fixed we then choose h sufficiently large so as to make (C 3) less than unity. Therefore the observed mean flow is indeed unstable (in the sense of Orr) to two-dimensional perturbations.

Appendix D.

In order to prove the inequality (5.12) we let

$$\overline{\phi_\xi \theta} = a(1-G(\zeta)),$$

where $a > 0$, and $G(\zeta)$ decreases monotonically from $G(0) = 1$ to $G(1) = 0$ [cf. (5.7)]. Since $s(\zeta)$ is assumed to be monotonic decreasing, the function

$$H(\zeta) \equiv \frac{s(\zeta)}{\int_0^1 s(\zeta)\, d\zeta} - 1$$

having zero average, also decreases monotonically as ζ increases. Therefore $H(\zeta) \geqslant 0$ for $\zeta \leqslant \zeta_p$ and $H(\zeta) \leqslant 0$ for $\zeta \geqslant \zeta_p$, where ζ_p is some value between zero and unity. Consequently

$$\int_0^1 GH\, d\zeta = \int_0^{\zeta_p} GH\, d\zeta + \int_{\zeta_p}^1 GH\, d\zeta \geqslant G(\zeta_p) \int_0^{\zeta_p} H\, d\zeta + G(\zeta_p) \int_{\zeta_p}^1 H\, d\zeta = 0$$

and it then follows that

$$\left[1 - \int_0^1 G\, d\zeta\right] - \left[1 - \frac{\int_0^1 Gs\, d\zeta}{\int_0^1 s\, d\zeta}\right] = \int_0^1 GH\, d\zeta \geqslant 0.$$

Both terms on the left-hand side of this relation are positive because $G \leqslant 1$, and therefore

$$\frac{1 - \int_0^1 Gs\, d\zeta \left(\int_0^1 s\, d\zeta\right)^{-1}}{1 - \int_0^1 G\, d\zeta} \leqslant 1,$$

or

$$\int_0^1 (1-G)s\, d\zeta \Big/ \int_0^1 (1-G)\, d\zeta \leqslant \int_0^1 s\, d\zeta,$$

or

$$\int_0^1 \overline{\phi_\xi \theta}\, s\, d\zeta \Big/ \int_0^1 \overline{\phi_\xi \theta}\, d\zeta \leqslant \int_0^1 s\, d\zeta.$$

REFERENCES

Busse, F. H. 1969 *Z. angew. Math. Phys.* **29**, 1–14.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

Deardorff, J. W. & Willis, G. E. 1967 *Quart. J. Roy. Met. Soc.* **93**, 166–175.

Gol'dshtik, M. A. 1969 *Sov. Phys. Dokl.* **13**, 1008–1010.

Hinze, J. O. 1959 *Turbulence*. McGraw-Hill.

Howard, L. N. 1972 *Ann. Rev. Fluid Mech.* **44**, 473–494.

Kline, S. J., Reynolds, W. C., Schraub, F. A. & Runstadler, P. W. 1967 *J. Fluid Mech.* **30**, 741–776.

Laufer, J. 1951 N.A.C.A. Rep. 1053.

LAUFER, J. 1954 N.A.C.A. Tech. Rep. 1174.

LIN, C. C. 1955 *Theory of Hydrodynamic Stability*. Cambridge University Press.

LINDEN, P. F. & SHIRTCLIFFE, T. G. L. 1978 *J. Fluid Mech.* **87**, 417–432.

MALKUS, W. V. R. 1956 *J. Fluid Mech.* **1**, 521–539.

MALKUS, W. V. R. 1963 Outline of a theory of turbulent convection. In *Theory and Fundamental Research in Heat Transfer*, pp. 203–212. Pergamon.

ORR, W. M. 1907 *Proc. Roy. Irish Academy* A **27**, 69–138.

REYNOLDS, W. C. & TIEDERMAN, W. G. 1967 *J. Fluid Mech.* **27**, 253–272.

TOWNSEND, A. A. 1959 *J. Fluid Mech.* **5**, 209–241.

TOWNSEND, A. A. 1976 *The Structure of Turbulent Shear Flow*. Cambridge University Press.